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Noether's theorem and dynamical groups in quantum mechanics

O Castaños, A Frank and R Lopez-Peña

Instituto de Ciencias Nucleares, Universidad Nacional Autonoma de Mexico, Circuito Exterior, CU, Apdo. Postal 70-543, 04510 Mexico DF, Mexico

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Abstract. In contrast with the case of symmetry groups, no unique definition for the dynamical group generators of a given quantum Hamiltonian has been adopted in the literature. Following some work by Malkin, Man'ko and Lipkin, Dothan proposed a general definition based on the symmetry group of the corresponding time-dependent quantum mechanical equation of motion for the system. We show that this definition arises naturally from the quantum version of time dependent Noether symmetry transformations and derive the dynamical algebras for some simple examples.

1. Introduction

Symmetry methods have been applied to the analysis of quantum mechanical systems in a wide variety of cases, ranging from exact solutions, for which the hydrogen atom [1] and the harmonic oscillator [2] are the best known examples, to models which incorporate in an approximate way the main features of the system under consideration [3-5]. The role of symmetry groups and their algebras is well understood and numerous applications have been carried out which explain features such as the degeneracy of states and the appearance of selection rules, often providing new insights into the nature of physical systems, besides simplifying the mathematical computations.

Non-invariance groups, whose generators do not all commute with the Hamiltonian of the system, have also been increasingly used in the literature, one example being the spectrum generating algebras [6] (notably SU(1, 1)), which provide an elegant means to determine the spectrum of certain second-order differential equations. Of greater importance are the so-called dynamical groups which, however, are often loosely defined. Thus, Wybourne writes: 'Ideally we seek a group that can yield the energy spectrum and the degeneracies of the levels, and that contains a set of operators that determine the transition probabilities between states' [6], while Wulfman defines it as a group that 'contains sufficient generators to enable one to formulate the dynamics of the system or set of systems solely in terms of operations on a single irreducible representation of the group' [7]. The latter definition is essentially equivalent to that introduced by Dothan *et al* in particle physics, who require that the Hamiltonian and other dynamical operators of the system be-expressible in terms of powers of the generators of the dynamical group [8]. We note that, in contrast to the well known prescription defining the elements of the symmetry algebra, they mainly constitute the maximum set of operators which close under commutation and commute with the Hamiltonian, no unique prescription for the dynamical group generators is provided by these definitions. The importance of dynamical groups (and their algebras) was

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first noted by Goshen and Lipkin [9] in the context of nuclear physics and later the method was exploited in particle physics by Barut and Bohm [10] and by Dothan *et al* [8]. Since then, there has been an enormous amount of applications, ranging from condensed matter physics to field theory and strings [11]. From the theoretical point of view, however, there remained a lack of a precise definition for this generalized symmetry. Malkin and Man'ko [12] pointed out that if $\psi(\boldsymbol{x}, t)$ is a solution of the time dependent Schrödinger equation

$$\left[\mathrm{i}\frac{\partial}{\partial t} - H(\boldsymbol{x}, \boldsymbol{p})\right]\psi(\boldsymbol{x}, t) = 0 \tag{1.1}$$

then $K(\boldsymbol{x}, \boldsymbol{p}, t)\psi(\boldsymbol{x}, t)$ is also a solution if K satisfies

$$\mathrm{i}\frac{\partial K}{\partial t} - [H, K] = 0 \tag{1.2}$$

that is, if K is a conserved quantity which in general depends explicitly on time. Lipkin [13] noted that if K satisfies (1.2) and in general $\partial K/\partial t \neq 0$, then $K\psi$ should be a linear combination of eigenstates of H with different energies so K generates the spectrum of H. Later on, Dothan [14] proposed to adopt (1.2) as a *definition* of the dynamical group generators. However, although a number of examples were discussed, no general technique to derive such generators was provided in [14]. We believe that the absence of such a procedure prevented this definition from becoming the standard one. Although some authors, notably Barut and Raczca [15], do refer to it, they do not attempt to *derive* the dynamical algebras from (1.2) but rather demonstrate the strength of the algebraic method once the dynamical algebra is known.

In a series of papers in recent years, a number of quantum mechanical systems have been shown to have exact solutions as a consequence of the existence of a dynamical supersymmetry [16]. This means that the Hamiltonian turns out to be proportional to a generator of a superalgebra and thus its spectrum can be obtained by algebraic manipulations. These examples are a generalization of the notion of spectrum generating algebras [6] to superalgebras. For our purposes, more than the particular examples analysed, we are interested in the technique proposed by Hooker *et al*, who generate their non-invariance (super) algebra using a classical Lagrangian and Noether's theorem, allowing time dependent variations [9], although their interest is restricted to the generation of the spectrum and not to the construction of the full dynamical (super) algebra. As is intuitively obvious, the introduction of such variations in a classical Lagrangian leads to conserved quantities whose Poisson bracket with the Hamiltonian is not null. That this set of conserved quantities close under the Poisson bracket operation is less obvious, however, and the fact that it does leads to the algebraic solutions for the system, once the quantum version of the algebra is constructed.

In this paper we fully exploit the procedure of [16] in order to show that the complete dynamical group of a stationary quantum system may be generated by this technique and that this constitutes an unambiguous definition, which arises naturally from the complete set of (in general time-dependent) constants of the motion of the system. We also show that these time-dependent charges can be simply interpreted as the generators of the symmetry group of the corresponding time-dependent quantum system. This result coincides with the definition of Dothan [14], equations (1.1) and (1.2). The method is illustrated in the next section by means of a simple one-dimensional

harmonic oscillator. In the last section we test the procedure by analysing the n-dimensional harmonic oscillator and the Coulomb system.

2. Noether's theorem and dynamical groups

We start by recalling Noether's theorem. Given a Lagrangian, which is in general a function of the coordinates, velocities and time

$$L = L(x_i, \dot{x}_i, t) \tag{2.1}$$

an arbitrary, infinitesimal transformation of the coordinate x_i

$$x_i \to x_i + \delta x_i \tag{2.2}$$

is a symmetry transformation if the corresponding variation in the Lagrangian induced by (2.2) can be written as a total time derivative

$$\delta L = \mathrm{d}\Omega/\mathrm{d}t \tag{2.3}$$

where the equality must hold independently of the equations of motion. Noether's theorem asserts that if (2.3) is satisfied, then to the symmetry transformation (2.2) corresponds a conserved quantity K (called Noether charge), given by

$$K = \sum \left(\frac{\partial L}{\partial \dot{x}_i}\right) \delta x_i - \Omega. \tag{2.4}$$

Noether's theorem is usually applied in a restricted form, involving only time-independent variations δx_i, since these give rise to energy preserving symmetries, i.e. conserved quantities whose Poisson bracket with the Hamiltonian is zero. Consider for example a general Lagrangian of the form $L = T - V(\boldsymbol{x})$, where T is the kinetic energy and V is the potential energy, which is assumed to depend only on the coordinates x_i. The variation

$$\delta x_i = \varepsilon \dot{x}_i \tag{2.5}$$

with an infinitesimal constant ε, induces the Lagrangian variation

$$\delta L = \varepsilon \frac{\mathrm{d}}{\mathrm{d}t}(T - V) \tag{2.6}$$

and hence the Noether charge is

$$K = \varepsilon(T + V) = \varepsilon H \tag{2.7}$$

which is proportional to the system's Hamiltonian. As another simple example, for a central potential $V(r)$ and the variation

$$\delta x_i = \varepsilon_{ijk} a_j x_k \tag{2.8}$$

where a_j is an infinitesimal constant vector, the Noether charge turns out to be

$$K = \boldsymbol{a} \cdot \boldsymbol{L} \tag{2.9}$$

where $\boldsymbol{L}$ is the angular momentum vector. Since $\boldsymbol{a}$ is arbitrary, we find that every component of $\boldsymbol{L}$ satisfies the equation

$$\frac{\mathrm{d}L_i}{\mathrm{d}t} = \{L_i, H\} = 0. \tag{2.10}$$

In addition, the L_i close under the Poisson bracket operation, i.e.

$$\{L_i, L_j\} = \varepsilon_{ijk} L_k. \tag{2.11}$$

In the general case, if K_1 and K_2 satisfy

$$\{K_1, H\} = \{K_2, H\} = 0 \tag{2.12}$$

one can prove that $\{\{K_1, K_2\}, H\} = 0$ using the Jacobi identity

$$\{H, \{K_1, K_2\}\} + \{K_1, \{K_2, H\}\} + \{K_2, \{H, K_1\}\} = 0. \tag{2.13}$$

Thus, the full set of Noether charges will close among themselves and form a Lie algebra [6] under the Poisson bracket operation.

The quantum mechanical version of (2.10)-(2.13) is obtained by the substitution [17]

$$\{A, B\} \to \frac{1}{\mathrm{i}} [\hat{A}, \hat{B}] \tag{2.14}$$

where [ ] denotes the commutator operation and the classical quantities A, B go to their operator form. For a central potential, we find the set of equations

$$\frac{\mathrm{d}\hat{L}_i}{\mathrm{d}t} = \frac{1}{\mathrm{i}} [\hat{L}_i, \hat{H}] = 0 \qquad [\hat{L}_i, \hat{L}_j] = \mathrm{i}\varepsilon_{ijk} \hat{L}_k \tag{2.15a, b}$$

which imply that the $\hat{L}_i$ form a Lie algebra under the commutator operation and generate the symmetry group SO(3) for the central system. Thus the quantum version of (2.12), (2.13) implies that one can start from the classical Lagrangian and use Noether's theorem (in the restricted sense of time-independent variations) to arrive at the quantum mechanical symmetry group of the system.

The above considerations are well known and were presented here for the sake of completeness. We now turn our attention to time-dependent variations. As mentioned in the introduction, it is clear from the start that such transformations will give rise to Noether charges that do not preserve the energy. In the classical case, the equation of motion for any function $A(\boldsymbol{x}, \boldsymbol{p}, t)$ in phase space is

$$\dot{A} \equiv \frac{\mathrm{d}A}{\mathrm{d}t} = \{A, H\} + \frac{\partial A}{\partial t} \tag{2.16}$$

and thus any constant of the motion K, will satisfy the equation $\dot{K} = 0$, or

$$\frac{\partial K}{\partial t} = -\{K, H\}. \tag{2.17}$$

The substitution (2.12) takes us again to the quantum case. Time-dependent variations will lead in general to time-dependent Noether charges and in this case (2.16) gives

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t} \{K_1, K_2\} &= \{\{K_1, K_2\}, H\} + \frac{\partial}{\partial t} \{K_1, K_2\} \\ &= -\left\{\frac{\partial K_1}{\partial t}, K_2\right\} - \left\{K_1, \frac{\partial K_2}{\partial t}\right\} + \frac{\partial}{\partial t} \{K_1, K_2\} = 0 \end{aligned} \tag{2.18}$$

where we used the Jacobi identity. This result is known as Poisson's theorem [18]. Thus, time-dependent Noether charges again satisfy the group property, in the generalized sense (2.18) (compare with (2.12), (2.13)). The closure clearly extends again to quantum mechanics through (2.14).

The question arises on what is the nature of this group and whether it may be useful, particularly in the quantum mechanical case. We show below that the set of time-dependent Noether charges are in fact simply related to the set of generators of the dynamical group of quantum systems.

The quantum mechanical description of the time-dependent wave equation for both relativistic and non-relativistic systems is given by [18]

$$\hat{H}\Phi = \mathrm{i}\frac{\partial \Phi}{\partial t} \tag{2.19}$$

where $\hat{H}$ denotes the Hamiltonian.

Assuming that there is a set $\{\hat{K}\}$ of constants of the motion, the time rate of change of any $\hat{K}$ in the Heisenberg picture is zero and thus, under the replacement of the Poisson bracket for its corresponding commutator, a relation analogous to (2.17) is satisfied:

$$-\mathrm{i}\frac{\partial \hat{K}}{\partial t} = [\hat{K}, \hat{H}]. \tag{2.20}$$

We next consider the transformation produced by the operator $\hat{K}$ on (2.19). By using (2.20) we arrive at the relation

$$\hat{H}(\hat{K}\Phi) = \mathrm{i}\frac{\partial}{\partial t}(\hat{K}\Phi) \tag{2.21}$$

which shows that one may interpret $\{\hat{K}\}$ as the set of transformations that leave invariant the time-dependent wave equation of quantum systems [14], i.e. if Φ satisfies (2.19), so does $\hat{K}\Phi$ [19].

To illustrate these results we consider a one-dimensional harmonic oscillator with the Lagrangian

$$L = \tfrac{1}{2}(\dot{x}^2 - x^2) \tag{2.22}$$

where units in which $m = w = 1$ are used.

A time-dependent transformation of x, to first order in the coordinates and velocities

$$\delta x = f(t)\dot{x} + g(t)x + h(t) \tag{2.23}$$

induces a variation of the Lagrangian given by

$$\begin{aligned}\delta L &= \left(\frac{\partial L}{\partial x}\right)\delta x + \left(\frac{\partial L}{\partial \dot{x}}\right)(\delta x)^{\cdot} \\ &= (-x)(f\dot{x} + gx + h) + \dot{x}(\dot{f}\dot{x} + f\ddot{x} + \dot{g}x + g\dot{x} + \dot{h}).\end{aligned} \tag{2.24}$$

This variation can be written in the form (2.3) with

$$\Omega = \tfrac{1}{2}f\dot{x}^2 + \tfrac{1}{2}(\dot{g} - \dot{f})x^2 + \dot{h}x \tag{2.25}$$

if the functions f, g and h satisfy the differential equations

$$\dot{f} = -2g \tag{2.26a}$$

$$\ddot{g} + 4g = 0 \tag{2.26b}$$

$$\ddot{h} + h = 0 \tag{2.26c}$$

which can be solved immediately to give

$$g = A\,e^{2it} + B\,e^{-2it} \tag{2.27a}$$

$$f = i(A\,e^{2it} - B\,e^{-2it}) + C \tag{2.27b}$$

$$h = D\,e^{it} + F\,e^{-it} \tag{2.27c}$$

where A, B, C, D and E are arbitrary constants. Substituting (2.24) and (2.26) into the corresponding expression (2.4) we get the following constants of the motion:

$$K_- = \left[\frac{i}{2}(p^2 - x^2) + xp\right] e^{2it} \tag{2.28a}$$

$$K_+ = \left[-\frac{i}{2}(p^2 - x^2) + xp\right] e^{-2it} \tag{2.28b}$$

$$K_0 = \tfrac{1}{2}(p^2 + x^2) \tag{2.28c}$$

$$F_- = (p - ix)\,e^{it} \tag{2.28d}$$

$$F_+ = (p + ix)\,e^{-it} \tag{2.28e}$$

where the relation $p = \partial L/\partial \dot{x} = \dot{x}$ has been used. Under the Poisson bracket operation the Noether charges satisfy the relations

$$\begin{aligned} &\{K_+, K_-\} = 4iK_0 \qquad \{K_0, K_\pm\} = \mp 2iK_\pm \\ &\{K_\pm, F_\pm\} = 0 \qquad \{K_\pm, F_\mp\} = 2F_\pm \end{aligned} \tag{2.29a}$$

$$\{K_0, F_\pm\} = \mp iF_\pm \tag{2.29b}$$

$$\{F_+, F_-\} = -2i. \tag{2.29c}$$

Note that these commutation relations are valid for all t, including $t = 0$. We readily identify these equations as defining a Lie algebra composed of a semidirect product of a one-dimensional Weyl algebra (2.28c) and a two-dimensional symplectic algebra (2.28a), $w(1) \wedge \mathrm{sp}(2, R)$. One may use equation (2.17) for the Hamiltonian $H = K_0$, to verify that the functions (2.27) are indeed conserved. The canonical quantization of these functions is accomplished in such a way as to preserve relations (2.28) after the substitution (2.14) has been carried out.

The $W(1) \wedge \mathrm{Sp}(2, R)$ group has been used in the literature as a dynamical group for the one-dimensional harmonic oscillator. Here we have proved that this group arises naturally from Noether's theorem as a *symmetry* group of the time-dependent system [14]. The realization (2.27) (in its quantum mechanical form) coincides with the standard realization for $t = 0$. Note that the time-independent generators are unitarily equivalent to the time-dependent ones: $\hat{K}(0) = e^{i\hat{H}t}\hat{K}(t)\,e^{-i\hat{H}t}$, so one may restrict the discussion to the former. In the next section we test the procedure for higher-dimensional systems.

3. Dynamical groups of quantum systems

In this section we consider two well known quantum mechanical systems for which the procedure indicated in the previous section to define the dynamical group is used. These are the n-dimensional harmonic oscillator and the Coulomb problems.

We start by considering the harmonic oscillator case. From the analysis made above the extension to this system is easily carried out by replacing the scalar functions (2.6) by the matrices f_{ij}, g_{ij} and h_i, with $i, j = 1, 2, \ldots, d$,

$$\delta x_i = f_{ij}(t)\dot{x}_j + g_{ij}(t)x_j + h_i. \tag{3.1}$$

The matrices f_{ij} and g_{ij} can be written as the sum of a symmetric matrix (f^s_{ij}, g^s_{ij}) and an antisymmetric matrix (f^a_{ij}, g^a_{ij}).

The change in the Lagrangian of a d-dimensional harmonic oscillator under the transformation (3.1) is written as a total time derivative if the following equations are satisfied:

$$\dot{f}^s_{ij} + 2g^s_{ij} = 0 \tag{3.2a}$$

$$\dot{g}^s_{ij} + 4g^s_{ij} = 0 \tag{3.2b}$$

$$\ddot{h}_i + h_i = 0 \tag{3.2c}$$

$$f^a_{ij} = 0 \tag{3.2d}$$

$$\dot{g}^a_{ij} = 0. \tag{3.2e}$$

The first three relations have the same form as (2.9). Thus the matrices f^s_{ij}, g^s_{ij} and h_i have the time dependence indicated in (2.10) and it is only necessary to make the replacement of the constants A, B, C, D and F by the constant matrices A_{ij}, B_{ij}, C_{ij}, D_i and F_i. Equation (3.2d) implies that the matrix f_{ij} must be symmetric, while (3.2e) indicates that g^a_{ij} is a constant matrix. The associated Noether charges are given by

$$K^+_{ij} = \left[-\frac{\mathrm{i}}{2}(p_i p_j - x_i x_j) + \frac{1}{2}(x_j p_i + x_i p_j) \right] \mathrm{e}^{-2\mathrm{i}t} \tag{3.3a}$$

$$K^0_{ij} = \tfrac{1}{2}(p_i p_j + x_i x_j) \tag{3.3b}$$

$$F^+_i = (p_i + \mathrm{i}x_i)\,\mathrm{e}^{-\mathrm{i}t} \tag{3.3c}$$

$$L_{ij} = (x_i p_j - x_j p_i) \tag{3.3d}$$

$$K^-_{ij} \equiv (K^+_{ij})^* \qquad F^-_i \equiv (F^+_i)^* \tag{3.3e}$$

where $*$ stands for complex conjugation. These functions of the dynamical variables close under the Poisson bracket operation and satisfy the relations

$$\{L_{ij}, L_{i'j'}\} = L_{jj'}\delta_{ii'} + L_{ii'}\delta_{jj'} + L_{i'i}\delta_{ij'} + L_{j'i}\delta_{i'j} \tag{3.4a}$$

$$\{L_{ij}, K^\tau_{i'j'}\} = K^\tau_{i'j}\delta_{ij'} + K^\tau_{jj'}\delta_{ii'} - K^\tau_{ii'}\delta_{jj'} - K^\tau_{ij'}\delta_{ji'} \tag{3.4b}$$

$$\{K^+_{ij}, K^-_{i'j'}\} = G_{i'i}\delta_{jj'} + G_{j'j}\delta_{i'i} + G_{ji'}\delta_{ij'} + G_{j'i}\delta_{ii'} \tag{3.4c}$$

$$\{K^0_{ij}, K^\sigma_{i'j'}\} = -\sigma\frac{\mathrm{i}}{2}\{K^\sigma_{ii'}\delta_{jj'} + K^\sigma_{ij'}\delta_{ii'} + K^\sigma_{i'j}\delta_{j'i} + K^\sigma_{jj'}\delta_{ii'}\} \tag{3.4d}$$

$$\{K^\tau_{ij}, K^\tau_{i'j'}\} = \{F^\sigma_i, K^\sigma_{i'j'}\} = \{F^\sigma_i, F^\sigma_j\} = 0 \tag{3.4e}$$

$$\{K^0_{ij}, F^\sigma_{i'}\} = -\sigma\mathrm{i}/2\{\delta_{i'j}F^\sigma_i + \delta_{ii'}F^\sigma_j\} \tag{3.4f}$$

$$\{K^\pm_{ij}, F^\pm_{i'}\} = F^\pm_i\delta_{jj'} + F^\pm_j\delta_{ii'} \tag{3.4g}$$

$$\{F^-_i, F^+_j\} = 2\mathrm{i}\delta_{ij} \tag{3.4h}$$

where the labels τ and σ take the values $+$, 0, $-$ and $+$, $-$ respectively. We also introduced the definition

$$G_{ij} = \tfrac{1}{2}L_{ij} - \mathrm{i}K^0_{ij}. \tag{3.5}$$

From the canonical quantization of (3.4) we conclude that the Noether charges (3.3) generate the semidirect product group $W(d) \wedge \mathrm{Sp}(2d, R)$, which is the dynamical symmetry group of the d-dimensional harmonic oscillator, which spans all states of the system in a single representation [7].

We now turn our attention to the Coulomb problem, in the formulation of Fock and Bargmann [1], where the Hamiltonian takes the form

$$H = \tfrac{1}{2}r(p^2+1) \tag{3.6}$$

also known as the pseudo-Coulomb Hamiltonian, whose bound-state symmetry group is the orthogonal group in four dimensions, SO(4). The group SO(4, 2) is usually considered as a dynamical group for (3.6), since it contains the symmetry group and additional generators that connect all bound states in the system [6]. However, it is important to note that an SO(4, 1) subgroup of SO(4, 2) already has a representation which spans all these states [20]. The generators of both groups connect all the eigenstates of (3.6), although SO(4, 1) does not include the dipole operator [6]. This is an example of the ambiguity in the definition of the dynamical group. In fact, other dynamical groups have been proposed for the Coulomb system [21]. We now apply Noether's theorem to Hamiltonian (3.6).

The corresponding Lagrangian for the pseudo-Coulomb Hamiltonian is given by

$$L = \frac{1}{2}\left(\frac{\dot{x}_k^2}{r} - r\right) \tag{3.7}$$

and in this case we can use Noether's inverse theorem to establish the associated symmetry transformations [18]. Since we know that SO(4) is the symmetry group for the system, we use the angular momentum and Runge-Lenz vectors to find

$$\delta x_l^L = \varepsilon_{ljk} a_j x_k \tag{3.8a}$$

$$\delta x_l^A = \frac{\boldsymbol{x}\cdot\boldsymbol{b}}{r}\dot{x}_l - \frac{\dot{\boldsymbol{x}}\cdot\boldsymbol{b}}{r}x_l - \frac{\boldsymbol{x}\cdot\dot{\boldsymbol{x}}}{r}b_l \tag{3.8b}$$

where a_j, b_l are infinitesimal constants and the indices l, j, k take the values 1, 2, 3. Besides these expressions, we know that a variation linear in $\dot{x}_k$ generates the Hamiltonian as conserved quantity.

Expressions (3.8) suggest that for the time-dependent variations we consider terms at most linear in the velocities and of the form:

$$\delta x_l = f_{lk}\dot{x}_k + g_{lk}x_k + \frac{(\boldsymbol{x}\cdot\boldsymbol{b})}{r}\dot{x}_l - \frac{\dot{\boldsymbol{x}}\cdot\boldsymbol{b}}{r}x_l - \frac{\boldsymbol{x}\cdot\dot{\boldsymbol{x}}}{r}b_l + rc_l \tag{3.9}$$

where f_{lk}, g_{lk}, b_l and c_l are time-dependent functions. This transformation induces a variation in the Lagrangian of the form (2.3), if the following equations are satisfied:

$$f_{lk} = f(t)\delta_{lk} \tag{3.10a}$$

$$g_{lk}^s = \delta_{lk}g(t) \qquad \dot{g}_{lk}^a = 0 \tag{3.10b}$$

$$\dot{f} = -g \tag{3.10c}$$

$$\ddot{g} + g = 0 \tag{3.10d}$$

$$\ddot{c}_l + c_l = 0 \tag{3.10e}$$

$$\dot{b}_l = c_l. \tag{3.10f}$$

The value of Ω is given by

$$\Omega = \frac{f}{2r}\dot{x}^2 + (\dot{g} - f/2)r + \boldsymbol{b}\cdot\boldsymbol{A} + (\dot{\boldsymbol{c}} + \boldsymbol{b})\cdot\boldsymbol{x} \tag{3.11a}$$

where A denotes the Runge-Lenz vector,

$$A_l = \frac{1}{2}\frac{\dot{x}^2}{r^2}x_l - \frac{\boldsymbol{x}\cdot\dot{\boldsymbol{x}}}{r^2}\dot{x}_l - \frac{1}{2}x_l. \tag{3.11b}$$

The solutions of (3.10) are straightforward and given by

$$g^a_{lk} = \varepsilon_{lkj}a_j \tag{3.12a}$$

$$g(t) = A\,e^{it} + B\,e^{-it} \tag{3.12b}$$

$$f(t) = i(A\,e^{it} - B\,e^{-it}) + C \tag{3.12c}$$

$$c_l(t) = D_l\,e^{it} + E_l\,e^{-it} \tag{3.12d}$$

$$b_l(t) = -i(D_l\,e^{it} - E_l\,e^{-it}) + F_l \tag{3.12e}$$

which contain fifteen independent constants, a_l, A, B, C, D_l, E_l, and F_l. The corresponding conserved quantities are given by

$$H = \tfrac{1}{2}r(p^2 + 1) \tag{3.13a}$$

$$C^+ = e^{-it}\{-iH + \boldsymbol{x}\cdot\boldsymbol{p} + ir\} \qquad C^- = (C^+)^* \tag{3.13b}$$

$$A_l = \tfrac{1}{2}p^2x_l - (\boldsymbol{x}\cdot\boldsymbol{p})p_l - \tfrac{1}{2}x_l \qquad L_l = \varepsilon_{lij}x_ip_j \tag{3.13c}$$

$$M_l^+ = e^{-it}\{iA_l + ix_l + rp_l\} \qquad M_l = (M_l^+)^* \tag{3.13d}$$

which are obtained by substituting (3.11) and (3.12) into the corresponding equation (2.4).

The Noether charges (3.13) close under the Poisson bracket operation and satisfy the relations

$$\{L_l, C^\pm\} = \{L_l, H\} = \{A_l, H\} = 0 \tag{3.14a}$$

$$\{L_l, L_k\} = \varepsilon_{lkj}L_j \qquad \{A_l, A_k\} = \varepsilon_{lkj}L_j \qquad \{L_l, A_k\} = \varepsilon_{lkj}A_j \tag{3.14b}$$

$$\{C^\pm, M_e^\pm\} = 0 \qquad \{L_l, M_k^\pm\} = \varepsilon_{lkj}M_j^\pm \tag{3.14c}$$

$$\{C^+, C^-\} = 2iH \qquad \{H, C^\pm\} = \mp iC^\pm \tag{3.14d}$$

$$\{A_l, C^\pm\} = \pm i\delta_{lk}M_l^\mp \tag{3.14e}$$

$$\{A_l, M_k^\pm\} = \pm i\delta_{lk}C^\pm \tag{3.14f}$$

$$\{M_l^+, M_k^-\} = 2\varepsilon_{lkj}L_j + 2i\delta_{lk}H. \tag{3.14g}$$

This Poisson bracket algebra corresponds to the SO(4, 2) Lie algebra [22] and expressions (3.14*b*) and (3.14*d*) to the subalgebras SO(4) and SO(2, 1), respectively. The canonical quantization of the Noether charges is carried out in such a way as to preserve relations (3.14) after the substitution (2.14) has been effected. We conclude that the symmetry group of the time-dependent pseudo-Coulomb Hamiltonian (3.6) is SO(4, 2). For $t = 0$ the commutation relations (3.14) are still satisfied, although the generators are not constants of the motion. We thus find that SO(4, 2) is the dynamical group for the stationary system.

4. Conclusions

In this paper we have shown that the dynamical group of a time-independent quantum system can be obtained by applying Noether's theorem to the classical Lagrangian, allowing time dependence in the infinitesimal parameters of the variation. The form of the latter is suggested by the time-independent variations, which lead to the Noether charges generating the symmetry group. Once the Poisson algebra has been constructed, canonical quantization is straightforward. The defining equation for the time-dependent generators $\hat{K}_i(t)$ is simply given by

$$\left[H - i\frac{\partial}{\partial t}, \hat{K}_i(t)\right] = 0 \tag{4.1}$$

which is equivalent to (2.21). The set $\{\hat{K}_i(t)\}$ thus generates the symmetry group for the time-dependent system, while the set $\{\hat{K}_i(0)\}$ generates the dynamical group for the corresponding stationary system [14]. This method leads to the $W(d) \wedge \mathrm{Sp}(2d, R)$ and SO(4, 2) dynamical groups for the d-dimensional harmonic oscillator and the Coulomb system, respectively. We plan to apply these ideas to other problems and explore their usefulness for more complex situations, such as those arising in connection with boson-fermion symmetries and supersymmetries [16].

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